N°. XX.

Dr. Rittenhouse, to Mr. Patterson, relative to a method of finding the sum of the several powers of the Sines, &c.

DEAR SIR,

Read May 18, 1792.

I Had discovered a very elegant theorem for determining the times of vibration of a pendulum in given arches of a circle; but it included a problem the solution of which I do not remember to have met with, though I cannot suppose that it has escaped the notice of mathematicians. It is, to find the sums of the several powers of the sines, either to a radius of unity or any other.

I was induced to attempt the means of doing this solely by its usefulness, but in prosecuting the enquiry I found much of that pleasing regularity, the discovery of which the geometrician often thinks a sufficient reward for his labours.

The sums of the odd powers of the sines bear a very simple relation to each other, and so do the sums of the even powers. But all the sums of the odd powers are incommensurable to all those of the even powers.

If we take the radius equal to unity the sum of all the sines, or their first powers, will be $= 1$, and the sum of all their squares $= \frac{1}{2}$ multiplied by the arch of 90°. The sum of all their cubes is $= \frac{2}{3}$, and the sum of their fourth powers $= \frac{3}{8}$ multiplied by the arch of 90°. The sum of the fifth powers is $= \frac{8}{15}$, and the sum of the 6th powers $= \frac{5}{16} \times$ by the arch of 90°.

I have not been able strictly to demonstrate any more than the two first cases. The others were investigated by the method of infinite series so far as to leave no doubt of

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the ultimate ratio which the ſum of the given power of the ſines bears to a known power of the radius.

Having proceeded ſo far as the 6th power the law of continuation became evident; ſo that, ſhould any problem in mathematical philoſophy require it, we may proceed as far as we pleaſe in ſumming the powers of the ſines. The law is this,

Make a fraction whoſe denominator is the index of the given power, and its numerator the ſame index, diminiſhed by unity, and multiplied by the ſquare of the radius; by this fraction multiply the ſum of the next but one lower power, and we have the ſum of the given power.

Thus 1ſt, the ſum of the 1ſt power of the ſines is $= rr$, or the ſquare of the radius 2d, ſum of the 2d, power or ſquares is $= \frac{1}{2} rr \times$ by the arch of 90°.	By Demonſtration.
3d, ſum of the 3d, power or cubes is $\frac{2}{3} rr$ of the 1ſt, or $= \frac{2}{3} r^4$. 4th. ſum of 4th powers is $= \frac{3}{4} rr$ of the 2d or $= \frac{3}{8} r^4$. $\times$ by the arch of 90°. 5th, ſum of 5th. powers is $= \frac{4}{5} rr$ of the 3d, or $= \frac{8}{15} r^6$. 6th, ſum of 6th. powers is $= \frac{5}{6} rr$ of the 4th or $= \frac{5}{16} r^6$. $\times$ by arch of 90°.	By Infinite Series.
7th, ſum of 7th. powers is $= \frac{6}{7} rr$ of the 5th, or $= \frac{16}{35} r^8$. 8th, ſum of 8th. powers is $= \frac{7}{8} rr$ of the 6th, or $= \frac{35}{128} r^3$. $\times$ by the arch of 90°.	By the Law of Continuation.

&c. &c.

Should your leiſure permit you to give any attention to this ſubject I ſhall be glad to ſee you furniſh a demonſtration for the 3d, or any ſubſequent caſe abovementioned.

I am, Sir,

Your moſt obedient humble ſervant,

DAVID RITTENHOUSE.

Index